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A new family of bivariate max-infinitely divisible distributions

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Abstract In this article we discuss the asymptotic behaviour of the componentwise maxima for a specific bivariate triangular array. Its components are given in terms of linear transformations of bivariate generalised symmetrised Dirichlet random vectors introduced in Fang and Fang (Statistical inference in elliptically contoured and related distributions. Allerton Press, New York, 1990). We show that the componentwise maxima of such triangular arrays is attracted by a bivariate max-infinitely divisible distribution function, provided that the associated random radius is in the Weibull max-domain of attraction.

Keywords Extremes of triangular arrays · Weibull max-domain of attraction · Max-infinitely divisible distribution · Weak convergence · Generalised symmetrised Dirichlet distributions · Asymptotically spherical random vectors

1 Introduction

Let (S_1, S_2) be a bivariate spherically symmetric random vector with almost surely positive random radius $R := \sqrt{S_1^2 + S_2^2}$. It is well-known (see e.g. Cambanis et al. 1981 or Fang et al. 1990) that

$$(S_1, S_2) \stackrel{d}{=} (R\cos(\Theta), R\sin(\Theta)), \quad \Theta \in (-\pi, \pi),$$

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where the random angle Θ is independent of R, and $\sin^2(\Theta)$ is Beta distributed with parameters 1/2,1/2 ($\stackrel{d}{=}$ means equality of distribution functions).

A natural generalisation of spherically symmetric random vectors introduced in Fang and Fang (1990) is a generalised symmetrised Dirichlet random vector $(S_1(a,b), S_2(a,b))$ with distribution function G and stochastic representation

$$(S_1(a,b), S_2(a,b)) \stackrel{d}{=} \left(R\cos(\Theta_{a,b}), R\sin(\Theta_{a,b})\right), \quad \Theta_{a,b} \in (-\pi, \pi),$$

where the random angle $\Theta_{a,b}$ is again independent of the associated random radius R, $\sin^2(\Theta_{a,b})$ is a Beta distributed with positive parameters a, b, and both $S_1(a,b), S_2(a,b)$ are symmetric about 0 satisfying the quadrant symmetry condition

$$\boldsymbol{P}\left\{(-1)^i S_1(a,b) > 0, (-1)^j S_2(a,b) > 0\right\} = 1/4, \quad i, j = 1, 2. \tag{1.1}$$

Let $(X_n^{(1)}, X_n^{(2)}), n \geq 1$ be independent bivariate random vectors with common distribution function G, and let $M_{in} := \max_{1 \leq j \leq n} X_j^{(i)}, i = 1, 2$ be the componentwise maxima. If the distribution function F of the associated random radius R is in the Gumbel or the Weibull max-domain of attraction, then in view of Proposition 3.4, 3.5 in Hashorva (2005b) there exit constants $a_n > 0, b_n$ such that the convergence in distribution

$$((M_{1n} - b_n)/a_n, (M_{2n} - b_n)/a_n) \stackrel{d}{\to} \mathcal{M}, \quad n \to \infty \tag{1.2}$$

holds with $\mathcal{M}$ a bivariate random vector with independent Gumbel or Weibull components, respectively. Hüsler and Reiss (1989) show that the limiting random vector $\mathcal{M}$ of the normalised maxima can have dependent components—which is of some interest for statistical modelling—if we consider the maxima of a triangular array.

A simple one can be introduced as follows: For $\rho_{in} \in (-1, 1], n \geq 1, i = 1, 2$ given constants define a triangular array of independent bivariate random vectors $\{(X_{jn}^{(1)}, X_{jn}^{(2)}), 1 \leq j \leq n, n \in \mathbb{N}\}$ via the stochastic representation

$$\begin{aligned}\left(X_{jn}^{(1)}, X_{jn}^{(2)}\right) \stackrel{d}{=} \Big(&\rho_{1n} S_1(a,b) + \sqrt{1-\rho_{1n}^2}\, S_2(a,b), \rho_{2n} S_1(a,b)\\ &+ \sqrt{1-\rho_{2n}^2}\, S_2(a,b)\Big), \quad 1 \leq j \leq n.\end{aligned} \tag{1.3}$$

If $\lim_{n\to\infty} \rho_{in} = 1, i = 1, 2$ then we have the convergence in probability $(n \to \infty)$

$$\left(X_{jn}^{(1)}, X_{jn}^{(2)}\right) \stackrel{p}{\to} (S_1(a,b), S_1(a,b)), \quad \forall j \geq 1.$$

If $M_{in} := \max_{1 \leq j \leq n} X_{jn}^{(i)}, n \geq 1, i = 1, 2$ is the componentwise maxima, then the above convergence in probability may eventually imply asymptotic dependence of the sample maxima, i.e. (1.2) holds where $\mathcal{M}$ has dependent components.

As shown in Hüsler and Reiss (1989) for the Gaussian case, a certain speed of convergence $\rho_{in} \to 1$ implies indeed asymptotic dependence of the components of maxima.

Extensions of the Gaussian model can be found in Gale (1980), Eddy and Gale (1981) and in the recent papers Hashorva (2006a,b,c) where both F in the Gumbel or in the Weibull max-domain of attraction are dealt with.

In the present paper we discuss the asymptotic behaviour of the triangular array defined in (1.3) assuming that the associated random radius R has distribution function F with upper endpoint 1 being in the Weibull max-domain of attraction. Explicitly we assume that

$$\lim_{n\to\infty} \sup_{x<0} \left| F^n(1+c_n x) - \exp(-|x|^\alpha) \right| = 0 \tag{1.4}$$

is satisfied where $\alpha > 0$ and $c_n := 1 - F^{-1}(1-1/n), n > 1$ with F^{-1} the generalised inverse of F. See Resnick (1987), Reiss (1989), Falk et al. (2004), or de Haan and Ferreira (2006) for further details on the max-domain of attractions.

In the main result of this contribution we show that the convergence in distribution in (1.2) holds with $a_n, n \geq 1$ some positive constants, $b_n := 1, n \geq 1$ and $\boldsymbol{\mathcal{M}}$ with dependent Weibull components, provided that $\rho_{in} \to 1, i = 1, 2$, with a certain speed (see below (2.6)).

Organisation of the paper: In Sect. 2 we present the main result. Its proof and related asymptotical results are relegated to Sect. 3 (last section).

2 Main result

Let $\{(X_{jn}^{(1)}, X_{jn}^{(2)}), 1 \leq j \leq n, n \in \mathbb{N}\}$ be a triangular array with stochastic representation (1.3). The aim of this section is to show under what conditions (1.2) holds for sample maxima of the triangular array of interests, and furthermore, to find the limiting distribution function of $\boldsymbol{\mathcal{M}}$.

In the elliptical setup ($a = b = 1/2$) we have in view of Lemma 12.1.2 in Berman (1992)

$$X_{jn}^{(1)} \stackrel{d}{=} X_{jn}^{(2)} \stackrel{d}{=} X_{11}^{(1)}, \quad 1 \leq j \leq n, n \geq 1.$$

Assuming that F is in the Weibull max-domain of attraction implies that the distribution function of $X_{11}^{(1)}$ is also in the Weibull max-domain of attraction (see Berman 1992; Hashorva 2006a). Hence the asymptotic behaviour of the components of the sample maxima is known for this situation. For the triangular array in (1.3) we have (a, b are positive constants)

$$X_{jn}^{(1)} \stackrel{d}{=} X_{1n}^{(1)}, \quad X_{jn}^{(2)} \stackrel{d}{=} X_{1n}^{(2)}, \quad 1 \leq j \leq n, n \geq 1.$$

The distribution function of $(X_{1n}^{(1)}, X_{1n}^{(2)}), n \geq 1$ depends in general on n if $\rho_{in}, i = 1, 2$ depends on n, hence for our general setup it is not clear what is the asymptotic behaviour of the components of the sample maxima. We show next that (1.4) still implies a similar asymptotic behaviour of the components of the sample maxima as in the elliptical setup.

For sake of simplicity we suppose that the upper endpoint of F is 1. Define further

$$I_{\alpha,a}(s,x,y) := \int_x^y (1-u^2)^\alpha \, |u+s|^{2a-1} \, du, \quad a>0, \alpha>0, s\ge 0, x<y, x,y\in\mathbb{R}$$

and

$$F_{a,b}(u) := 1-(1-u)^a(1-F(u))2^{a-1}\Gamma(a+b)/(\Gamma(a)\Gamma(b)), \\ a>0, b>0, u\in(0,1), \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function. Write in the sequel $I_{\alpha,a}(s), s\ge 0$ instead of $I_{\alpha,a}(s,-1,1)$ and define further the family of distribution functions on $[-1,1]$ by

$$\Upsilon_{\alpha,a}(s,y) := \mathbf{1}(y\in[-1,1])\frac{I_{\alpha,a}(s,-1,y))}{I_{\alpha,a}(s)}, \quad a>0, \alpha>0, s\ge 0, y\in\mathbb{R}, \tag{2.2}$$

with $\mathbf{1}(\cdot\in[-1,1])$ the indicator function and set $\Upsilon_{\alpha,a}(s,y) := 1$ if $y>1$.

The function $F_{a,b}$ with generalised inverse $F_{a,b}^{-1}$ plays an important role for the definition of the constants entering in the asymptotics, namely we define

$$r_n := 1-F_{a,b}^{-1}(1-1/n), \quad n\to\infty. \tag{2.3}$$

We state now the main result:

Theorem 2.1 *Let $\{(X_{jn}^{(1)}, X_{jn}^{(2)}), 1\le j\le n, n\in\mathbb{N}\}$ be a triangular array of independent bivariate random vectors with underlying distribution function G_n satisfying (1.3) with a, b positive constants and $\rho_{in}\in[0,1], n\ge 1, i=1,2$. Let further $u_n\in(0,1), n\in\mathbb{N}$ be given constants converging to 1 as $n\to\infty$ such that*

$$\lim_{n\to\infty}\frac{1-\rho_{in}}{1-u_n} = \rho_i^2 < \infty, \rho_i \ge 0, \quad i=1,2. \tag{2.4}$$

If the distribution function F of R fulfills (1.4) with $\alpha>0$ and $F(0)=0$, then we have for $i=1,2$

$$\boldsymbol{P}\left\{X_{1n}^{(i)} > u_n\right\} = (1+o(1))[1-F_{a,b}(u_n)]I_{\alpha,a}(\rho_i), \quad n\to\infty, \tag{2.5}$$

with $F_{a,b}(u)$ defined in (2.1). If for $i=1,2$ and $r_n, n\ge 1$ as in (2.3)

$$\lim_{n\to\infty}\frac{1-\rho_{in}}{r_n} = 2\delta_i^2 \tag{2.6}$$

holds with δ_1, δ_2 *two non-negative constants such that* $\gamma := \delta_2 - \delta_1 > 0$, *then we have*

$$\lim_{n\to\infty} \sup_{x<0, y<0} \left| G_n^n(1 + r_n x, 1 + r_n y) - H_{\alpha,a,\delta_1,\delta_2}(x, y) \right| = 0, \tag{2.7}$$

where

$$\begin{aligned} H_{\alpha,a,\delta_1,\delta_2}&(x, y) \\ = \exp\Bigg(&- |x|^{\alpha+a} I_{\alpha,a}\left(\frac{2\delta_1}{\sqrt{2|x|}}\right) \Upsilon_{\alpha,a}\left(\frac{2\delta_1}{\sqrt{2|x|}}, \frac{\frac{y-x}{2\gamma} + \gamma}{\sqrt{2|x|}}\right) \\ &-|y|^{\alpha+a} I_{\alpha,a}\left(\frac{2\delta_2}{\sqrt{2|y|}}\right)\left[1-\Upsilon_{\alpha,a}\left(\frac{2\delta_2}{\sqrt{2|y|}}, \frac{\frac{y-x}{2\gamma} - \gamma}{\sqrt{2|y|}}\right)\right]\Bigg), \quad x, y < 0. \end{aligned} \tag{2.8}$$

Remark 1. For $a = 1/2$ and $c \geq 0$ we have

$$I_{\alpha,1/2}(c) = I_{\alpha,1/2}(0) = 2\int_0^1 (1 - u^2)^\alpha \, du = \frac{\Gamma(\alpha + 1)\sqrt{\pi}}{\Gamma(\alpha + 3/2)},$$

hence the marginal distributions of $H_{\alpha,1/2,\delta_1,\delta_2}$ are Weibull. Further

$$1 - \Upsilon_{\alpha,1/2}(s, x) = \Upsilon_{\alpha,1/2}(s, -x) = \Upsilon_{\alpha,1/2}(0, -x), \quad \forall x \in [-1, 1], s \geq 0,$$

thus $H_{\alpha,1/2,\delta_1,\delta_2}$ reduces (after scaling) to the bivariate distribution function introduced in Hashorva (2005a), which initially appears in the context of the extremes of convex hulls in Gale (1980) and Eddy and Gale (1981).

2. The distribution function $H_{\alpha,a,\delta_1,\delta_2}$ is a max-id. distribution. See Resnick (1987) and Falk et al. (2004) for details on max-id. distributions.
   The marginal distributions of $H_{\alpha,a,\delta_1,\delta_2}$ are Weibull only for $a = 1/2$, hence $H_{\alpha,a,\delta_1,\delta_2}$ is not a max-stable distribution function for $a \neq 1/2$. This is the case also for $a = 1/2$, which follows easily since the max-stability requires (see Falk et al. 2004)

$$H_{\alpha,a,\delta_1,\delta_2}(t^{-1/(\alpha+a)}x, t^{-1/(\alpha+a)}y) = (H_{\alpha,a,\delta_1,\delta_2}(x, y))^{1/t}, \quad \forall x, y, -t < 0.$$

   The above condition is not satisfied for all $x, y, -t$ negative.
3. In Theorem 12.3.3 of Berman (1992) the asymptotic relation in (2.5) is shown for $\rho_{in} = 1, 1 \leq i \leq n, n \geq 1$.
4. It follows easily that $F_{a,b}$ is a monotone function, hence the asymptotic solution r_n that satisfies (2.3) exists.
5. In the context of the extremes of convex hulls Gale (1980) and Eddy and Gale (1981) derive assuming F possesses a density function f with algebraic tail the same asymptotic distribution as in Hashorva (2006a). This fact has been kindly noted by one Referee of the paper.

Using the stochastic representation of $(S_1(a,b), S_2(a,b))$ examples of bivariate generalised symmetrised Dirichlet arrays satisfying the conditions of our main theorem above can be easily constructed by choosing R so that its distribution function F is in the Weibull max-domain of attraction. Based on extreme value theory several known distribution functions are possible candidate for F, for instance the Beta distribution function. We present next an illustrating example where the starting point is the density function of $(S_1(a,b), S_2(a,b))$.

Example 1 (Kummer–Beta Dirichlet Distribution) Define a bivariate generalised symmetrised Dirichlet distribution (see Fang and Fang 1990; Kotz et al. 2000 for details on Dirichlet distribution) with density function

$$h(x,y) := c_1 g_{\alpha,\beta,\lambda}(x^2+y^2)\,|x|^{2a-1}\,|y|^{2b-1},$$
$$x,y\in\mathbb{R}, x^2+y^2\le 1, \quad a,b>0, a+b+\beta>1, c_1>0$$

where

$$g_{\alpha,\beta,\lambda}(r) = (1-r)^{\alpha-1} r^{\beta-1}\exp(-\lambda r), \quad 0<r<1, \alpha>0, \beta>0, \lambda\ge 0.$$

Let $(S_1(a,b), S_2(a,b))$ be a random vector with density function h, which we refer to as a Kummer-Beta random vector. It follows that the associated random radius R with distribution function F has density function (see Hashorva et al. 2007)

$$f(x) = c_2(1-x^2)^{\alpha-1}x^{2(a+b+\beta-1)-1}\exp(-\lambda x^2), \quad \forall x\in(0,1),$$

with $c_2>0$ a norming constant. It follows that F is in the max-domain of attraction of Ψ_α, hence our theorem above is applicable for this example.

3 Further results and proof

We give first two lemmas needed for the proof of the main result below.

Lemma 3.1 *Let $\mu, \mu_n, n\ge 1$ be positive finite measures defined on the interval $[a,b], 0\le a<b<\infty$, and let $f, f_n, n\ge 1$ be a sequence of positive measurable functions. Assume that the weak convergence*

$$\mu_n \overset{w}{\to} \mu, \quad n\to\infty \tag{3.1}$$

holds. Suppose further that $f_n, n\ge 1$ are uniformly bounded on $[a,b]$. If for any sequence $x_n, n\ge 1, x_n\in[a,b]$ such that $x_n\to x\in\Delta$ we have $\lim_{n\to\infty} f_n(x_n) = f(x)$ with Δ a Borel set satisfying $\mu(\Delta)=1$, then we have

$$\lim_{n\to\infty}\int_a^b f_n(s)\,\mu_n(ds) = \int_a^b f(s)\,\mu(ds) < \infty. \tag{3.2}$$

Proof We have $\mu([a,b]) > 0$ and $\mu_n([a,b]) > 0$ for all $n \geq 1$. Define a new sequence of probability measures on $[a,b]$ by $\mu_n^*(\cdot) := \mu_n(\cdot)/\mu_n([a,b]), n \geq 1, \mu^*(\cdot) := \mu(\cdot)/\mu([a,b])$. By the assumptions we have the weak convergence $\mu_n^* \overset{w}{\to} \mu^*, n \to \infty$. Next, applying Theorem 3.27 of Kallenberg (1997) we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_a^b f_n(s)\,\mu_n(ds) &= \mu([a,b]) \lim_{n\to\infty} \int_a^b f_n(s)\,\mu_n^*(ds)\\ &= \mu([a,b]) \int_a^b f(s)\,\mu^*(ds)\\ &= \int_a^b f(s)\,\mu(ds) < \infty,\end{aligned}$$

hence the proof is complete. □

Lemma 3.2 *Let* (S_1, S_2) *be a bivariate random vector with almost surely positive random radius* $R = \sqrt{S_1^2 + S_2^2}$. *Let further* (U, V) *be a bivariate random vector with stochastic representation*

$$(U,V) \overset{d}{=} \left(\rho_1 S_1 + \sqrt{1-\rho_1^2}S_2,\, \rho_2 S_1 + \sqrt{1-\rho_2^2}S_2\right),$$

where $\rho_1 > \rho_2, \rho_i \in [0,1], i = 1,2$. *Define* $p_{u,v} := \boldsymbol{P}\{U > u, V > v\}$ *with* u, v *positive constants. If* $p_{u,v} > 0$, *then we have*

$$\begin{aligned}p_{u,v} = \boldsymbol{P}&\left\{R > \frac{u}{\cos(\Theta - \arccos(\rho_1))}, \Theta \in \left[\beta_{\rho_1,\rho_2}(u,v), \frac{\pi}{2} + \arccos(\rho_1)\right), \Theta\right\}\\ +\boldsymbol{P}&\left\{R > \frac{v}{\cos(\Theta - \arccos(\rho_2))}, \Theta \in \left[-\frac{\pi}{2} + \arccos(\rho_2), \beta_{\rho_1,\rho_2}(u,v)\right)\right\},\end{aligned} \tag{3.3}$$

where

$$\Theta := \arccos(S_1/R), \quad \textit{and}$$
$$\beta_{\rho_1,\rho_2}(u,v) := \arccos(\rho_1) + \arctan\left(\frac{v/u - \cos(\arccos(\rho_2) - \arccos(\rho_1))}{\sin(\arccos(\rho_2) - \arccos(\rho_1))}\right).$$

Redefine $\beta_{\rho_1,\rho_2}(u,v) := \arccos(\rho_2) - \pi/2$ *if* $\beta(\rho_1,\rho_2) < \arccos(\rho_2) - \pi/2$ *and* $\beta_{\rho_1,\rho_2}(u,v) = \arccos(\rho_1) + \pi/2$ *if* $> \arccos(\rho_1) + \pi/2$.

Proof Since the random radius R is almost surely positive, then the random angle $\Theta := \arccos(S_1/R)$ is well-defined. Further, $S_1^2/R^2 + S_2^2/R^2 = 1$ holds almost surely, hence we may write

$$(U, V) \stackrel{d}{=} R(\cos(\Theta - z_1), \cos(\Theta - z_2)),$$

with $z_i := \arccos(\rho_i) \in [0, \pi/2], i = 1, 2$ and $z_1 < z_2$. For any $u, v \in (0, \infty)$ we have

$$\begin{aligned}
&\boldsymbol{P}\{U > u, V > v\}\\
&\quad = \boldsymbol{P}\{R\cos(\Theta - z_1) > u, R\cos(\Theta - z_2) > v, \Theta - z_1, \Theta - z_2 \in [-\pi/2, \pi/2]\}\\
&\quad = \boldsymbol{P}\{R\cos(\Theta - z_1) > u, R\cos(\Theta - z_2) > v, z_2 - \pi/2 \le \Theta \le z_1 + \pi/2\}\\
&\quad = \boldsymbol{P}\left\{R > \max\left(\frac{u}{\cos(\Theta - z_1)}, \frac{v}{\cos(\Theta - z_2)}\right), z_2 - \pi/2 \le \Theta \le z_1 + \pi/2\right\}\\
&\quad = \boldsymbol{P}\left\{R > \frac{u}{\cos(\Theta - z_1)}, \Theta \in [\beta_{\rho_1,\rho_2}(u, v), \frac{\pi}{2} + z_1)\right\}\\
&\qquad + \boldsymbol{P}\left\{R > \frac{v}{\cos(\Theta - z_2)}, \Theta \in [-\frac{\pi}{2} + z_2, \beta_{\rho_1,\rho_2}(u, v))\right\},
\end{aligned}$$

with $\beta_{\rho_1,\rho_2}(u, v)$ the solution of (recall z_i depends on $\rho_i, i = 1, 2$)

$$\frac{\cos(\beta - z_1)}{\cos(\beta - z_2)} = \frac{u}{v}.$$

We have that

$$\beta_{\rho_1,\rho_2}(u, v) = z_1 + \arctan\left(\frac{v/u - \cos(z_2 - z_1)}{\sin(z_2 - z_1)}\right).$$

Set $\beta_{\rho_1,\rho_2}(u, v) := z_2 - \pi/2$ if $\beta < z_2 - \pi/2$ and $\beta_{\rho_1,\rho_2}(u, v) := z_1 + \pi/2$ if $\beta > z_1 + \pi/2$, hence the proof is complete. □

Proof of Theorem 2.1 Let Q denote the distribution function of $\Theta_{a,b}$ and put $z_{in} := \arccos(\rho_{in}), i = 1, 2, n \in \mathbb{N}$. Clearly, $z_{in} \in [0, \pi]$ and furthermore (1.3) implies

$$\begin{aligned}
\left(X_{jn}^{(1)}, X_{jn}^{(2)}\right) &\stackrel{d}{=} \left(\rho_{1n} S_1(a, b) + \sqrt{1 - \rho_{1n}^2} S_2(a, b), \rho_{2n} S_1(a, b) + \sqrt{1 - \rho_{2n}^2} S_2(a, b)\right)\\
&\stackrel{d}{=} \left(R\cos(\Theta_{a,b} - z_{1n}), R\cos(\Theta_{a,b} - z_{2n})\right), \quad 1 \le j \le n, n \in \mathbb{N}.
\end{aligned} \tag{3.4}$$

Recall that $S_i(a, b), i = 1, 2$ in our definition are symmetric about 0. Since $R > 0$ almost surely being further independent of the random angle $\Theta_{a,b}$, we may write for

n large and $i = 1, 2$

$$
\begin{aligned}
\boldsymbol{P}\left\{X_{1n}^{(i)} \geq u_n\right\} &= \boldsymbol{P}\left\{R\cos(\Theta_{a,b} - z_{in}) > u_n, \Theta_{a,b} - z_{in} \in [-\pi/2, \pi/2]\right\} \\
&= \int_{z_{in}}^{\pi/2+z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta) \\
&\quad + \int_{-\pi/2+z_{in}}^{z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta). \qquad (3.5)
\end{aligned}
$$

We consider next the first integral above. Indeed, the asymptotic behaviour of that integral for $z_{in} = 0, n \in \mathbb{N}$ follows by Theorem 12.3.3 of Berman (1992). Since F has upper endpoint 1, we have for all n large

$$
\begin{aligned}
&\int_{z_{in}}^{\pi/2+z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta) \\
&= \int_{z_{in}}^{\min(\pi/2+z_{in}, \psi_n+z_{in})} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta),
\end{aligned}
$$

with $\psi_n := \arccos(u_n), n \geq 1$. By the assumptions

$$
\lim_{n\to\infty} \psi_n = 0, \quad \text{and} \quad \lim_{n\to\infty} z_{in} = 0, \quad i = 1, 2,
$$

hence we obtain for large n using further the quadrant symmetry condition (1.1) and the fact that $\psi_n \geq 0$

$$
\begin{aligned}
&\int_{z_{in}}^{\pi/2+z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta) \\
&= \frac{1}{4} \int_{\sin^2(z_{in})}^{\sin^2(\psi_n+z_{in})} [1 - F(u_n/\cos(\arccos((1-y)^{1/2}) - z_{in}))]\, dB(y, a, b) \\
&= \frac{1 - F_{a,b}(u_n)}{2} \int_{\frac{\sin^2(z_{in})}{2(1-u_n)}}^{\frac{\sin^2(\psi_n+z_{in})}{2(1-u_n)}} \frac{1 - F(u_n/\cos(\arccos((1-2(1-u_n)y)^{1/2}) - z_{in}))}{1 - F(u_n)} \\
&\quad \times d\mu_n(y), \qquad (3.6)
\end{aligned}
$$

with

$$F_{a,b}(u) := 1 - (1-u)^a (1 - F(u)) 2^{a-1} \Gamma(a+b)/(\Gamma(a)\Gamma(b)), \quad u \in (0,1)$$

and $\mu_n, n \ge 1$ a sequence of positive finite measure defined by

$$\mu_n((x,y]) := H(y, 1-u_n) - H(x, 1-u_n), \quad 0 \le x < y < \infty,$$

where $H(s,z) = (2z)^{-a} B(2sz, a, b)\Gamma(a)\Gamma(b)/\Gamma(a+b), s, z > 0$ and $B(s,a,b)$ is the Beta distribution function with positive parameters a, b. Condition (2.4) and the fact that $\lim_{n\to\infty} u_n = 1$ yield

$$\lim_{n\to\infty} \frac{\psi_n}{\sqrt{2(1-u_n)}} = \lim_{n\to\infty} \frac{\arccos(u_n)}{\sqrt{2(1-u_n)}} = 1, \quad \lim_{n\to\infty} \frac{z_{in}}{\sqrt{2(1-u_n)}} = \rho_i, \quad i = 1, 2,$$

hence

$$\lim_{n\to\infty} \frac{\sin^2(\psi_n + z_{in})}{2(1-u_n)} = (1+\rho_i)^2, \quad \lim_{n\to\infty} \frac{\sin^2(z_{in})}{2(1-u_n)} = \rho_i^2.$$

Further, (1.4) implies

$$\lim_{t\downarrow 0} \frac{1 - F(1-ct)}{1 - F(1-t)} = c^\alpha, \quad \forall c > 0,$$

hence (2.4) yields for any $y_n \to y, n \to \infty$ with $\rho_i^2 < y < (1+\rho_i)^2$

$$\begin{aligned} &\lim_{n\to\infty} \frac{1 - F(u_n/\cos(\arccos((1-2(1-u_n)y_n)^{1/2}) - z_{in}))}{1 - F(u_n)} \\ &\quad = \lim_{n\to\infty} \frac{1 - F(u_n + (\sqrt{y_n} - \rho_i)^2(1-u_n))}{1 - F(u_n)} = (1 - (\sqrt{y} - \rho_i)^2)^\alpha. \end{aligned}$$

It follows easily that for any $0 \le x < y < \infty$

$$\lim_{n\to\infty} (H(y, 1-u_n) - H(x, 1-u_n)) = (y^a - x^a)/a,$$

consequently we obtain the weak convergence $\mu_n \xrightarrow{w} \mu, n \to \infty$, with μ a positive finite measure defined by

$$\mu((x,y]) := (y^a - x^a)/a, \quad 0 \le x < y < \infty.$$

Applying Lemma 3.1 we obtain

$$
\begin{aligned}
(3.6) &= (1+o(1))\frac{1-F_{a,b}(u_n)}{2}\int_{\rho_i^2}^{(1+\rho_i)^2}(1-(\sqrt{y}-\rho_i)^2)^{\alpha}y^{a-1}\,dy\\
&= (1+o(1))[1-F_{a,b}(u_n)]\int_{\rho_i}^{1+\rho_i}(1-(s-\rho_i)^2)^{\alpha}s^{2a-1}\,ds, \quad n\to\infty.
\end{aligned}
$$

We deal next with the second integral in (3.5). If $\rho_i > 1, i = 1, 2$ then for all large n we have $z_{in} - \psi_n \geq 0$. The fact that F has upper endpoint equal 1 implies for all n large

$$
\int_{-\pi/2+z_{in}}^{z_{in}}[1-F(u_n/\cos(\theta-z_{in}))]\,dQ(\theta) = \int_{z_{in}-\psi_n}^{z_{in}}[1-F(u_n/\cos(\theta-z_{in}))]\,dQ(\theta).
$$

Recall $\lim_{n\to\infty}(z_{in}-\psi_n) = 0$. With similar arguments as above we obtain as $n \to \infty$

$$
\begin{aligned}
&\int_{-\pi/2+z_{in}}^{z_{in}}[1-F(u_n/\cos(\theta-z_{in}))]\,dQ(\theta)\\
&= \int_{z_{in}-\psi_n}^{z_{in}}[1-F(u_n/\cos(\theta-z_{in}))]\,dB(\theta,a,b)\\
&= (1+o(1))\frac{1-F_{a,b}(u_n)}{2}\\
&\quad\times\int_{\frac{\sin^2(z_{in}-\psi_n)}{2(1-u_n)}}^{\frac{\sin^2(z_{in})}{2(1-u_n)}}\frac{[1-F(u_n/\cos(\arccos((1-2(1-u_n)y)^{1/2})-z_{in}))]}{1-F(u_n)}\,d\mu_n(y)\\
&= (1+o(1))[1-F_{a,b}(u_n)]\frac{1}{2}\int_{(1-\rho_i)^2}^{\rho_i^2}(1-(\sqrt{y}-\rho_i)^2)^{\alpha}y^{a-1}\,dy\\
&= (1+o(1))[1-F_{a,b}(u_n)]\int_{\rho_i-1}^{\rho_i}(1-(s-\rho_i)^2)^{\alpha}s^{2a-1}\,ds.
\end{aligned}
$$

If $\rho_i < 1, i = 1, 2$ we have for n large

$$\int_{-\pi/2+z_{in}}^{z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta)$$
$$= \int_0^{z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta) + \int_{z_{in}-\psi_n}^{0} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta).$$

As above we obtain

$$\int_0^{z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta)$$
$$= (1 + o(1))[1 - F_{a,b}(u_n)] \int_0^{\rho_i} (1 - (s - \rho_i)^2)^\alpha s^{2a-1}\, ds, \quad n \to \infty,$$

and

$$\int_{z_{in}-\psi_n}^{0} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta)$$
$$= \int_0^{\psi_n - z_{in}} [1 - F(u_n/\cos(\theta + z_{in}))]\, dQ(\theta)$$
$$= \frac{1}{4} \int_0^{\psi_n - z_{in}} [1 - F(u_n/\cos(\theta + z_{in}))]\, dB(\theta, a, b)$$
$$= (1 + o(1))[1 - F_{a,b}(u_n)] \int_0^{1-\rho_i} (1 - (s + \rho_i)^2)^\alpha s^{2a-1}\, ds$$
$$= (1 + o(1))[1 - F_{a,b}(u_n)] \int_{\rho_i - 1}^{0} (1 - (s - \rho_i)^2)^\alpha\, |s|^{2a-1}\, ds, \quad n \to \infty.$$

The case $\rho_i = 1$ follows with similar arguments. Consequently, for any $\rho_i \ge 0$ we have

$$\int_{-\pi/2+z_{in}}^{z_{in}} [1 - F(u_n/\cos(\theta - z_{in}))]\, dQ(\theta)$$

$$= (1 + o(1))[1 - F_{a,b}(u_n)] \int_{\rho_i - 1}^{\rho_i} (1 - (s - \rho_i)^2)^\alpha \, |s|^{2a-1} \, ds, \quad n \to \infty.$$

Putting together we obtain

$$\boldsymbol{P}\left\{X_{1n}^{(i)} > u_n\right\} = (1 + o(1))[1 - F_{a,b}(u_n)] \int_{\rho_i - 1}^{1+\rho_i} (1 - (s - \rho_i)^2)^\alpha \, |s|^{2a-1} \, ds$$

$$= (1 + o(1))[1 - F_{a,b}(u_n)] \int_{-1}^{1} (1 - s^2)^\alpha \, |s + \rho_i|^{2a-1} \, ds, \quad n \to \infty,$$

thus the first claim follows.

Next, define r_n for all large n by $r_n := 1 - F_{a,b}^{-1}(1 - 1/n)$ with $F_{a,b}^{-1}$ the generalised inverse of $F_{a,b}$. For any $x < 0$ (substituting $u_n := u_n(x) = 1 - r_n \, |x|$ above) we have

$$n\boldsymbol{P}\left\{X_{1n}^{(i)} > 1 + r_n x\right\} = (1 + o(1)) \, |x|^{\alpha+a} \, I_{\alpha,a}\left(\frac{2\delta_i}{\sqrt{2\,|x|}}\right), \quad n \to \infty. \quad (3.7)$$

Let in the following $x, y \in (-\infty, 0)$ be fixed and define for all large n

$$\beta_{x,y,n} := z_{1n} + \arctan\left(\frac{(1 + r_n y)/(1 + r_n x) - \cos(z_{2n} - z_{1n})}{\sin(z_{2n} - z_{1n})}\right), \quad n \geq 1$$

and

$$A_1(x) := \frac{1 + r_n}{\cos(\Theta_{a,b} - z_{1n})}, \quad A_2(y) := \frac{1 + r_n}{\cos(\Theta_{a,b} - z_{2n})}.$$

We write for simplicity in the following β_n instead of $\beta_{x,y,n}$. We have

$$\lim_{n\to\infty} \beta_n r_n^{-1/2} = \lim_{n\to\infty} \left[z_{1n} + \arctan\left(\frac{\frac{1+r_n y}{1+r_n x} - \cos(z_{2n} - z_{1n})}{\sin(z_{2n} - z_{1n})}\right)\right] r_n^{-1/2}$$

$$= \frac{y - x}{2(\delta_2 - \delta_1)} + \delta_2 + \delta_1,$$

hence for $u_n = 1 + r_n x, n \geq 1$ we obtain

$$\lim_{n\to\infty} \frac{\beta_n}{\sqrt{1 - u_n^2}} = \lim_{n\to\infty} \frac{\beta_n}{\sqrt{2 r_n \, |x|}} = \left(\frac{y - x}{2(\delta_2 - \delta_1)} + \delta_2 + \delta_1\right) \frac{1}{\sqrt{2\,|x|}}.$$

Since $\lim_{n\to\infty} 1 + r_n s = 1$ for all $s < 0$ and (2.6) implies $z_{1n} < z_{2n}$ for all large n we obtain applying Lemma 3.2 for x, y negative and n large

$$\begin{aligned}
&\boldsymbol{P}\left\{R\cos(\Theta_{a,b}-z_{1n}) > 1+r_n x,\, R\cos(\Theta_{a,b}-z_{2n}) > 1+r_n y\right\}\\
&\quad = \boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b} \in [\beta_n, \frac{\pi}{2}+z_{1n})\right\}\\
&\qquad + \boldsymbol{P}\left\{R > A_2(y),\, \Theta_{a,b} \in [-\frac{\pi}{2}+z_{2n}, \beta_n)\right\}. \qquad (3.8)
\end{aligned}$$

We assume for simplicity that both probabilities above are strictly positive for all large n. If $\beta_n \ge 0$ for all large n, then using the fact that $\Theta_{a,b}$ is independent of R we have with similar arguments for any $x < 0$ as $n \to \infty$

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b} \in \left[\beta_n, \frac{\pi}{2}+z_{in}\right]\right\}\\
&\quad = \lim_{n\to\infty}\left[n\boldsymbol{P}\left\{X_{1n}^{(1)} > 1+r_n x\right\}\right] \frac{\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b}\in[\beta_n, \frac{\pi}{2}+z_{in}]\right\}}{\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b}\in[\frac{-\pi}{2}, \frac{\pi}{2}]\right\}}\\
&\quad = |x|^{\alpha+a} \int_{\left(\frac{y-x}{2(\delta_2-\delta_1)}+\delta_2+\delta_1\right)/\sqrt{2|x|}}^{1+\delta_1\sqrt{2/|x|}} \left(1-\left(s-\delta_1\sqrt{2/|x|}\right)^2\right)^{\alpha} |s|^{2a-1}\, ds\\
&\quad = |x|^{\alpha+a} \int_{\left(\frac{y-x}{2(\delta_2-\delta_1)}+\delta_2-\delta_1\right)/\sqrt{2|x|}}^{1} (1-s^2)^{\alpha} \left|s+2\delta_1/\sqrt{2|x|)}\right|^{2a-1} ds\\
&\quad = |x|^{\alpha+a} I_{\alpha,a}\left(\frac{2\delta_1}{\sqrt{2|x|}}\right)\left[1-\Upsilon_{\alpha,a}\left(\frac{2\delta_1}{\sqrt{2|x|}}, \frac{\frac{y-x}{2\gamma}+\gamma}{\sqrt{2|x|}}\right)\right],
\end{aligned}$$

with $\gamma := \delta_2 - \delta_1$ and $\Upsilon_{\alpha,a}$ as in (2.2). If $\liminf_{n\to\infty} \beta_n r_n^{-1/2} < 0$ we have

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b}\in[\beta_n, \frac{\pi}{2}+z_{in}]\right\}\\
&\quad = \lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b}\in[0, \frac{\pi}{2}+z_{in}]\right\}\\
&\qquad + \lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b}\in[\beta_n, 0]\right\}.
\end{aligned}$$

For the first term on the right hand side above we obtain

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b}\in[0, \frac{\pi}{2}+z_{in}]\right\}\\
&\quad = |x|^{\alpha+a} \int_0^{1+\delta_1\sqrt{2/|x|}} \left(1-\left(s-\delta_1\sqrt{2/|x|}\right)^2\right)^{\alpha} |s|^{2a-1}\, ds,
\end{aligned}$$

whereas for the second term we have

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b} \in [\beta_n, 0]\right\} \\
&= \lim_{n\to\infty} \frac{n}{4} \int_0^{-\beta_n} [1 - F((1 + r_n x)/\cos(\theta - z_{1n}))]\, dB(\theta, a, b) \\
&= |x|^{\alpha+a} \int_0^{\left(\frac{x-y}{2(\delta_2-\delta_1)}-\delta_2-\delta_1\right)/\sqrt{2|x|}} (1 - (s + \delta_1\sqrt{2/|x|})^2)^\alpha\, |s|^{2a-1}\, ds \\
&= |x|^{\alpha+a} \int_{\left(\frac{y-x}{2(\delta_2-\delta_1)}+\delta_2+\delta_1\right)/\sqrt{2|x|}}^{0} (1 - (s - \delta_1\sqrt{2/|x|})^2)^\alpha\, |s|^{2a-1}\, ds.
\end{aligned}$$

Hence we obtain again

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_1(x),\, \Theta_{a,b} \in \left[\beta_n, \frac{\pi}{2} + z_{in}\right]\right\} \\
&= |x|^{\alpha+a}\, I_{\alpha,a}\left(\frac{2\delta_1}{\sqrt{2\,|x|}}\right)\left[1 - \Upsilon_{\alpha,a}\left(\frac{2\delta_1}{\sqrt{2\,|x|}}, \frac{\frac{y-x}{2\gamma} + \gamma}{\sqrt{2\,|x|}}\right)\right].
\end{aligned}$$

We consider now the second term in (3.5). Assume for simplicity that $\liminf_{n\to\infty} \beta_n < 0$. By the assumptions we have for any y negative

$$\begin{aligned}
&\lim_{n\to\infty} n\boldsymbol{P}\left\{R > A_2(y),\, \Theta_{a,b} \in \left[-\frac{\pi}{2} + z_{2n}, \beta_n\right)\right\} \\
&= \lim_{n\to\infty} \frac{n}{4} \int_{-\beta_n}^{\pi/2 - z_{2n}} [1 - F((1 + r_n y)/\cos(\theta + z_{2n}))]\, dB(\theta, a, b) \\
&= |y|^{\alpha+a} \int_{\left(\frac{x-y}{2(\delta_2-\delta_1)}-\delta_2-\delta_1\right)/\sqrt{2|y|}}^{1-\delta_2\sqrt{2/|y|}} \left(1 - \left(s + \delta_2\sqrt{2/|y|}\right)^2\right)^\alpha |s|^{2a-1}\, ds \\
&= |y|^{\alpha+a} \int_{\delta_2\sqrt{2/|y|}-1}^{\left(\frac{y-x}{2(\delta_2-\delta_1)}+\delta_2+\delta_1\right)/\sqrt{2|y|}} \left(1 - \left(s - \delta_2\sqrt{2/|y|}\right)^2\right)^\alpha |s|^{2a-1}\, ds \\
&= |y|^{\alpha+a} \int_{-1}^{\left(\frac{y-x}{2\gamma}-\gamma\right)/\sqrt{2|y|}} (1 - s^2)^\alpha \left|s + \delta_2\sqrt{2/|y|}\right|^{2a-1}\, ds
\end{aligned}$$

$$= |y|^{\alpha+a}\, I_{\alpha,a}\left(\frac{2\delta_2}{\sqrt{2\,|y|}}\right)\Upsilon_{\alpha,a}\left(\frac{2\delta_2}{\sqrt{2\,|y|}}, \frac{\frac{y-x}{2\gamma}-\gamma}{\sqrt{2\,|y|}}\right).$$

Again, the case $\beta_n \geq 0, n \geq 1$ can be shown with similar arguments. Hence the proof follows easily using further (3.7). □

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